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ON THE IMPROBABILITY OF FINDING ISOLATED SHOALS IN THE OPEN SEA BY SAILING OVER THE GEOGRAPHICAL POSITIONS IN WHICH THEY ARE CHARTED.

By Mr. G. W. Littlehales, Washington, D. C.

Many of the isolated shoals that are represented on nautical charts of the oceans have been located from the reports of mariners who have discovered them incidentally in making voyages of commerce. Previous to the year 1860, when there was no exact knowledge of the depths of the oceans, the vague reports of navigators, often doubtless based upon the observation of floating objects and of misleading appearances of the surface of the sea, caused the charting of many dangers for the existence of which there is no substantial foundation. But, as our knowledge of bathymetry increased, the existence of many of them was disproved, and they were removed from the charts.

As a result of these experiences, there arose a traditional distrust among mariners and hydrographers of the existence of many of these dangers that still appear on the charts with well founded evidence, and there is perhaps a disposition on the part of many to claim that they should be removed upon scant evidence of their non-existence. It is not uncommon for a mariner to report that, being in the vicinity of a charted rock or shoal, he laid his course so as to pass over the geographical position assigned to it with one hundred fathoms of line out or with lookouts posted aloft, but was unable to detect any evidence of its existence, and that he does not believe, therefore, that the rock or shoal exists.

It seems necessary, therefore, to inquire into the degree of confidence that can be placed in such a piece of evidence of the non-existence of a danger, and to establish what probability there would be of finding it under these conditions.

Suppose that A discovers, in the open ocean, a shoal r miles in radius, and determines the geographical position of its center subject to extreme errors of m miles in longitude and n miles in latitude; and that B, who is able to establish his geographical position within the same limits of extreme error as A, attempts to find the shoal again by proceeding to the geographical position assigned to it by A. What is the probability that he will find it?

If A, after making the discovery, had revisited the shoal a great number of times and had deduced the latitude and longitude of the same spot, under the same circumstances, at each visit, the latitudes would all differ from the

true latitude, and, likewise, the longitude from the true longitude. If we call the differences between the true latitude and the deduced latitudes errors of latitude, and lay them off, according to their signs, to the right and left of an assumed origin, and then, corresponding to each error as an abscissa, erect an ordinate of a length proportional to the probability of that error, these ordinates and abscissas will be the coordinates of the probability curve. And, likewise, if the errors in longitude were found and plotted in conjunction with their probabilities, a similar curve would be developed.

In this investigation the probability curve, ordinarily represented by Laplace's formula, $y = ce^{-a^2t^2}$, will be replaced by two equally inclined straight lines AB and AB' as shown in figure 1.

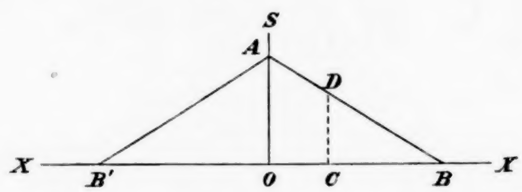


FIGURE 1.

This substitution, which has been employed by Hélie in his Traité de Balistique Expérimentale and referred to by Wright in his work on the Adjustment of Observations, causes an appreciable but extremely small error which has no practical significance when we consider that, from the nature of the calculations about to be made, absolute precision is not to be sought.

The probability of having an error between $OC = x$ and $x + \Delta x$ (figure 1) to the right of the axis OS is equal to $s\Delta x$. As, in this case, OB and OB' measure the extreme errors, all possible errors are comprised between zero and OB, and zero and OB'; and the sum of all the elements which are singly represented by $s \,.\, dx$, or the area of the triangle ABB', should be equal to unity which is the measure of certainty. The equation to the straight line AB will be, calling m the extreme error OB and b the intercept on the axis of S,

$$\frac{s}{b} + \frac{x}{m} = 1\,.$$

But, since the area $ABB' = b \times m = 1$ or $b = \dfrac{1}{m}$, this equation becomes

$$sm + \frac{x}{m} = 1\,,$$

or

$$s = \frac{m - x}{m^2}\,.$$

And since x can only vary between zero and m, the probability of having an error between x and $x + \Delta x$ will be :

$$p_1 = \frac{m - x}{m^2} \Delta x . \tag{1}$$

The causes which produce the grouping of a number of deduced geographical positions around the true one are of two kinds ; one tending to place the deduced latitude to the north or south of the true latitude, and the other tending to place the deduced longitude to the east or west of the true longitude. So that a particular deduced geographical position P will be the result of having an error OA in latitude and an error OB in longitude.

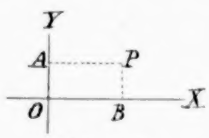


FIGURE 2.

The probability that the geographical position deduced by A, upon his discovery of the shoal, occupies a certain position with reference to the true geographical position of the shoal is, therefore, easily deduced. Through the true geographical position of the shoal let two rectangular axes, OX and OY, be passed as shown in figure 2. Upon the former conceive errors in longitude to be measured, and upon the latter, errors in latitude. The position P, of which the coordinates are x and y, results from the concurrence of two conditions, the error of x miles in longitude and the error of y miles in latitude. The probability p_1 of an error between x and $x + \Delta x$ is, as shown by equation (1),

$$p_1 = \frac{m - x}{m^2} \Delta x ;$$

and, in the same manner, the probability p_2 of an error between y and $y + \Delta y$ will be

$$p_2 = \frac{n - y}{n^2} \Delta y . \tag{2}$$

In these formulas, m and n represent respectively the extreme errors in longitude and latitude in miles.

The probability p of having, at the same time, the error x and the error y, or of deducing the geographical position P as the position of the shoal, will be the product $p_1 p_2$, or

$$p = \frac{(m - x)(n - y)}{m^2 n^2} \Delta x \, \Delta y, \tag{3}$$

an equation in which x can vary from zero to m, and y from zero to n. It is, therefore, applicable to the first right angle of the axes OX and OY, but, in order to make it applicable to other quadrants, it is only necessary to change the signs of x and y.

Equation (3) then expresses the probability that A's determination of the geographical position of the shoal is in error by x miles in longitude and y miles in latitude.

If the center of the shoal were really located in the geographical position assigned to it by A, and B should succeed in coming within r miles of it, he would find the shoal since its radius is r miles.

We have, therefore, as the second step in the solution of the problem, to determine what is the probability that B will come within a circular area, r miles in radius, having its center anywhere within the rectangle described about the true position of the shoal with sides equal to the extreme errors to which the determinations of latitude and longitude by A and B are subject.

To find the probability, P, of coming within any portion of the rectangle of extreme errors inclosed by a curve whose equation is $y = f(x)$, it is sufficient to integrate the expression (3) between limits depending only upon $y = f(x)$, and we shall have, in the first right angle,

$$P = \frac{1}{m^2n^2}\int dx \int (m - x)(n - y)\,dy. \tag{4}$$

For a circular area of radius r, we shall have for the first quadrant,

$$P = \frac{1}{m^2n^2}\int_0^r (m - x)\,dx \int_0^{\sqrt{r^2 - x^2}} (n - y)\,dy\,;$$

and for the whole circle,

$$P = \frac{4}{m^2n^2}\int_0^r (m - x)\,dx \int_0^{\sqrt{r^2 - x^2}} (n - y)\,dy\,,$$

or

$$P = \frac{2r^2}{mn}\left[\frac{\pi}{2} - \frac{2r}{3m} - \frac{2r}{3n} + \frac{r^2}{4mn}\right]. \tag{5}$$

The probability that B will find the shoal depends upon the concurrence of two independent conditions whose separate probabilities are represented by equations (3) and (5), respectively, and is, therefore, equal to $P.\,p$, or

$$\frac{2r^2}{mn}\left\{\frac{\pi}{2} - \frac{2r}{3m} - \frac{2r}{3n} + \frac{r^2}{4mn}\right\}\frac{(m - x)(n - y)}{m^2n^2}\,\Delta x\Delta y\,.$$

Integrating the two expressions which make up equation (3) between the limits x and $x + \Delta x$ and y and $y + \Delta y$, respectively, the above expression becomes:

$$\frac{2r^2}{mn}\left\{\frac{\pi}{2} = \frac{2r}{3m} - \frac{2r}{3n} + \frac{r^2}{4mn}\right\}\frac{\Delta x}{m}\left[1 - \frac{2x + \Delta x}{2m}\right]\frac{\Delta y}{n}\left[1 - \frac{2y + \Delta y}{2n}\right],$$

which, for $r = 1$ mile, $x = 2$ miles, $y = 2$ miles, $m = 10$ miles, $n = 10$ miles, and Δx and Δy each equal to 1 mile, becomes $\frac{1}{6173}$. That is, under the conditions stated, B would stand one chance in 6173 of finding the shoal.

U. S. Hydrographic Office, *Nov.* 12, 1895.

NOTE ON THE CONGRUENCE $2^{4n} \equiv (-)^n (2n)!/(n!)^2$, WHERE $2n+1$ IS A PRIME.

By Prof. F. Morley, Haverford, Pa.

1. There are two ways of integrating $\cos^{2n+1} x dx$, one by a Fourier series, where we express $\cos^{2n+1} x$ in terms of cosines of multiples of x, the other by a "formula of reduction." Taking the integral from $x = 0$ to $x = \pi/2$, and equating the two forms of the integral, we have an algebraic identity, which will, on a suitable supposition as to the integer n, yield a theorem as to prime numbers, trivial or otherwise. Doing this, we have, first,

$$2^{2n} \cos^{2n+1} x = \cos(2n+1)x + (2n+1)\cos(2n-1)x$$
$$+ \frac{(2n+1).2n}{1.2}\cos(2n-3)x + \ldots + \frac{(2n+1)2n\ldots(n+2)}{n!}\cos x,$$

$$2^{2n}\int_0 \cos^{2n+1} x dx = \frac{\sin(2n+1)x}{2n+1} + \frac{2n+1}{2n-1}\sin(2n-1)x + \ldots,$$

$$2^{2n}\int_0^{\frac{1}{2}\pi} \cos^{2n+1} x dx = (-)^n\left[\frac{1}{2n+1} - \frac{2n+1}{2n-1} + \ldots\right];$$

and second, from the formula of reduction,

$$\int_0^{\frac{1}{2}\pi} \cos^{2n+1} x dx = \frac{2n(2n-2)\ldots 2}{(2n+1)(2n-1)\ldots 3}; \qquad (1)$$

so that the algebraic identity is

$$2^{2n}\frac{2n(2n-2)\ldots 2}{(2n+1)(2n-1)\ldots 3}$$
$$=(-)^n\left[\frac{1}{2n+1} - \frac{2n+1}{2n-1} + \frac{(2n+1)2n}{1.2(2n-3)} - \ldots + (-)^n\frac{(2n+1)2n\ldots(n+2)}{n!}\right].$$

Let $2n+1$ be a prime p. Let us use the notation $a/b \equiv 0$, mod c, where a/b is a *fraction*, to mean that when the fraction is in its lowest terms the numerator has the factor c. Then, from the above identity,

$$2^{2n}\frac{2n(2n-2)\ldots 2}{(2n+1)(2n-1)\ldots 3} - (-)^n\frac{1}{2n+1} \equiv 0, \bmod p,$$

or

$$2^{2n}\frac{2n(2n-2)\ldots 2}{(2n-1)(2n-3)\ldots 1} - (-)^n \equiv 0, \bmod p^2,$$

or

$$2^{4n} \frac{(n!)^2}{(2n)!} - (-)^n \equiv 0 \, , \text{ mod } p^2 ,$$

or

$$2^{4n} - (-)^n \frac{(2n)!}{(n!)^2} \equiv 0 \, , \text{ mod } p^2 , \tag{2}$$

the left hand member being of course an integer.

This result is given in Mathews, Theory of Numbers, p. 318, Ex. 16.

When $n = 1, 2, 3$, the left hand member of (2) is respectively 18, 250, 4116, that is $2 \cdot 3^2$, $2 \cdot 5^3$, $2^2 \cdot 3 \cdot 7^3$. Thus, when $p = 5$ and 7, the left hand member $\equiv 0$, mod p^3, not merely mod p^2. I have to prove that this is so when p is a prime > 3.

2. It is convenient to prove, first, that when $p > 3$

$$\frac{1}{1^2} + \frac{1}{3^2} + \ldots + \frac{1}{(p-2)^2} \equiv 0 \, , \text{ mod } p . \tag{3}$$

Using the notation and results of Chrystal, Algebra, Vol. ii, p. 525, let

$$(x+1)(x+2)\ldots(x+p-1) = x^{p-1} + A_1 x^{p-2} + \ldots + A_{p-1} .$$

We have

$$[(p-1)!]^2 \left[\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{(p-1)^2} \right] = A^2_{p-2} - 2A_{p-1}A_{p-3} ;$$

or since $A_{p-2}, A_{p-3} \equiv 0$ mod p, if $p > 3$,

$$\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{(p-1)^2} \equiv 0 \text{ mod } p . \tag{4}$$

Now (writing always $2n + 1 = p$),

$$\frac{1}{1^2} - \frac{1}{(2n)^2} + \frac{1}{2^2} - \frac{1}{(2n-1)^2} + \ldots + \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

$$= \frac{(2n)^2 - 1^2}{1^2(2n)^2} + \frac{(2n-1)^2 - 2^2}{2^2(2n-1)^2} + \ldots + \frac{(n+1)^2 - n^2}{n^2(n+1)^2} \equiv 0 \text{ mod } p ,$$

for each numerator has, and each denominator has not, the factor $2n + 1$.

Hence, both sum and difference of the expressions

$$\frac{1}{1^2} + \frac{1}{2^2} + \ldots + \frac{1}{n^2} \quad \text{and} \quad \frac{1}{(n+1)^2} + \ldots + \frac{1}{(2n)^2}$$

$\equiv 0$, mod p; hence, also, each expression $\equiv 0$ mod p. Hence, again,

$$\frac{1}{2^2} + \frac{1}{4^2} + \ldots + \frac{1}{(2n)^2} \equiv 0 \, , \text{ mod } p .$$

Hence we can remove the even terms from the statement (4), and there remains (3).

3. Instead of expressing $\cos^p x$ as a sum of cosines, let us take the formula which expresses $\cos px$ as a power-series in $\cos x$. This is shown in treatises on trigonometry to be

$$(-)^n \cos px = p \cos x - \frac{p(p^2-1^2)}{3!}\cos^3 x + \frac{p(p^2-1^2)(p^2-3^2)}{5!}\cos^5 x - \ldots$$
$$+ (-)^n 2^{p-1} \cos^p x .$$

Multiply by dx and integrate from 0 to $\frac{1}{2}\pi$; then, using (1),

$$\frac{1}{p} = p - \frac{p(p^2-1^2)}{3!}\frac{2}{3} + \frac{p(p^2-1^2)(p^2-3^2)}{5!}\frac{2.4}{3.5} + \cdots$$
$$+ (-)^n 2^{p-1}\frac{2.4\ldots(p-1)}{3.5\ldots p}.$$

Therefore,

$$\frac{1}{p} - (-)^n 2^{p-1}\frac{2.4\ldots(p-1)}{3.5\ldots p} \equiv p\left\{1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots + \frac{1}{(p-2^2)^2}\right\}, \bmod p^3$$

i. e. (Art. 2) $\qquad \equiv 0, \bmod p^2$.

Therefore

$$(-)^n 2^{p-1}\frac{2.4\ldots(p-1)}{1.3\ldots(p-2)} - 1 \equiv 0, \bmod p^3$$

whence, as in Art. 1,

$$2^{4n} - (-)^n \frac{(2n)!}{(n!)^2} \equiv 0, \bmod p^3,$$

where $2n+1$ is a prime p, greater than 3.

HAVERFORD COLLEGE, *Nov.* 5, 1895.

EQUATIONS AND VARIABLES ASSOCIATED WITH THE LINEAR DIFFERENTIAL EQUATION.

By Dr. Geo. F. Metzler, Kingston, Ont., Canada.

My attention was first drawn to this subject by reading a memoir by A. R. Forsyth, published in the Phil. Trans. of the Royal Society, Vol. 179 (1888), pp. 377–489, and entitled Invariants, Covariants, and Quotient-derivatives associated with Linear Differential Equations.

Also, reference may be made to Professor Craig's Treatise, Chapter XIII; or my dissertation, entitled Invariants and Equations associated with the Linear Differential Equation.

As much is gained through the adoption of a convenient and brief form of notation, I introduce the following:

$$\frac{dy}{dx} \equiv y' , \quad \frac{d^n y}{dx^n} = y^{(n)} , \quad \text{etc.},$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \equiv (y_1 y_2') = (1\,2'),$$

$$\begin{vmatrix} y_1, & y_2, & y_3, & \ldots, & y_k, & \ldots, & y_n \\ y_1', & y_2', & y_3', & \ldots, & y_k', & \ldots, & y_n' \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)}, & y_2^{(n-1)}, & y_3^{(n-1)}, & \ldots, & y_k^{(n-1)}, & \ldots, & y_n^{(n-1)} \end{vmatrix} \equiv (y_1 y_2' y_3'' \ldots y_k^{(k-1)} \ldots y_n^{(n-1)}) \\ \equiv (1\,2'\,3'' \ldots k^{(k-1)} \ldots n^{(n-1)}).$$

I.—Associate Equations.

A linear differential equation of the nth order

$$A = 0 = y^{(n)} + \frac{n\,.\,(n-1)}{1\,.\,2} P_2 y^{(n-2)} + \ldots + \frac{n!}{r!\,(n-)\,r!} P_r y^{(n-r)} + \ldots + P_n y$$

has n fundamental solutions $y_1, y_2, \ldots, y_n$. From any two of these y_α, y_β we can form the function $(y_\alpha y_\beta' - y_\alpha' y_\beta)$, or in the abbreviated notation $(\alpha\beta')$, and from the n y's $\frac{n(n-1)}{2!}$ such functions, in general independent, which may be taken as the solutions of a linear differential equation of the $\frac{n(n-1)}{1\,.\,2}$th order

$$A_1 = 0 .$$

Taking any three of the solutions y, we can form functions $(y_\alpha y_\beta' y_\gamma'')$ or $(\alpha\beta'\gamma'')$, $\frac{n(n-1)(n-2)}{3!}$ in number, in general independent, and thus forming the solutions of a linear differential equation of the $\frac{n(n-1)(n-2)}{3!}$th order

$$A_2 = 0 .$$

Similarly, from the combination of four solutions y we obtain $\frac{n!}{4!(n-4)!}$ functions $(\alpha\beta'\gamma''\delta''')$ which are the solutions of a linear differential equation

$$A_3 = 0 .$$

Proceeding thus, and combining the y's five at a time, we obtain the solutions of $A_4 = 0$ of order $\frac{n!}{5!(n-5)!}$; combining them six at a time, we form the solutions of $A_5 = 0$ of order $\frac{n!}{6!(n-6)!}$; finally, combining the y's $n-1$ at a time, n functions are obtained solutions of a linear differential equation of the nth order

$$A_{n-2} = 0 .$$

These equations $A_1 = 0$, $A_2 = 0$, $A_3 = 0, \ldots, A_{n-2} = 0$ are called, respectively, the first, second, third, ..., $(n-2)$th associate equations of the original equation $A = 0$.

The last, viz, $A_{n-2} = 0$ has long been known as the adjoint of $A = 0$, or Lagrange's adjoint equation. Let $u_1, u_2, \ldots, u_n$, denote its solutions, where

$$u_\alpha = (-1)^{\alpha-1} \begin{vmatrix} y_1, & y_2, & \ldots, & y_{\alpha-1}, & y_{\alpha+1}, & \ldots, & y_n \\ y'_1, & y'_2, & \ldots, & y'_{\alpha-1}, & y'_{\alpha+1}, & \ldots, & y'_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-2)}, & y_2^{(n-2)}, & \ldots, & y_{\alpha-1}^{(n-2)}, & y_{\alpha+1}^{(n-2)}, & \ldots, & y_n^{(n-2)} \end{vmatrix}$$

$$\equiv (-1)^{\alpha-1}(y_1 y'_2 \cdots y_{\alpha-1}^{(\alpha-2)} y_{\alpha+1}^{(\alpha-1)} \cdots y_n^{(n-2)}).$$

When we consider the determinant

$$\begin{vmatrix} y_1, & y_2, & y_3, & \ldots, & y_n \\ y'_1, & y'_2 & y'_3, & \ldots, & y'_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-2)}, & y_2^{(n-2)}, & y_3^{(n-2)}, & \ldots, & y_n^{n-2} \\ y_1^{(l)}, & y_2^{(l)}, & y_3^{(l)}, & \ldots, & y_n^{(l)} \end{vmatrix}$$

we see that it vanishes for all values of l from 0 to $n-2$, and as the first $n-1$ rows give the solutions of the adjoint $A_{n-2}=0$, we have

$$y_1^{(l)}u_1+y_2^{(l)}u_2+y_3^{(l)}u_3+\ldots+y_n^{(l)}u_n=0\,,\quad (l=0,1,\ldots,n-2)\,. \tag{1}$$

The determinant also vanishes when we replace the u's by their derivatives $u_1^{(k)}\;u_2^{(k)}\ldots u_n^{(k)}$ provided $k+l \gtrless n-1$. Thus we have the more general relation

$$y_1^{(k)}u_1^{(l)}+y_2^{(k)}u_2^{(l)}+\ldots+y_n^{(k)}u_n^{(l)}=0,\;(k=0,1,\ldots,n-1;\,l=0,1,\ldots,n-1) \tag{2}$$

provided $k+l \gtrless n-1$.

These relations between the solutions of an equation and its adjoint become more interesting when the equation is self-adjoint, i. e. when $A_{n-2}=0$ is the same equation as $A=0$.

In the year 1889 I pointed out to Professor Craig and Professor Forsyth the following relation between the associate equations: The kth associate of the adjoint equation is the same as the $(n-2-k)$th associate of the original equation. This is proven in my dissertation and is included in a theorem due to Clebsch, published in the "Abhandlungen der Kön. Gesellschaft der Wissenschaften zu Göttingen, Band XVII," Ueber die Fundamentalaufgabe der Invarianttheorie.

II.—The Self-Adjoint Equation.

When $A_{n-2}=0$ is the same equation as $A=0$ the u must be linear functions of the y, thus:

$$u_i=\sum_1^n a_{ik}y_k\,,\qquad (i=1,2,\ldots,n) \tag{3}$$

where the a_{ik} are constants.

The condition necessary and sufficient that $A=0$ should be self-adjoint is that the invariants with odd suffix vanish. This theorem found in an article by Professor Brioschi published 1891 was proven by myself two years earlier (see Professor Craig's Treatise, p. 495).

If in (1) we let $l=0$ and then substitute the u from (3) we obtain

$$\varphi_{(y)}=\sum_1^n\sum_1^n a_{ik}y_iy_k=0\,.$$

To consider more closely the constants a_{ki}, let $n=3$; then $A=0$ becomes

$$y'''+3P_2y'+P_3y=0\,,$$

where

$$3P_2=-\frac{(2\;3''')}{(2\;3')}\text{ and }P_3=\frac{(2'\;3''')}{(2\;3')}\,,$$

and

$$\varphi(y) = a_1 y_1^2 + (a_{12} + a_{21}) y_1 y_2 + (a_{13} + a_{31}) y_1 y_3 + a_{22} y_2^2 + (a_{23} + a_{32}) y_2 y_3 + a_{33} y_3^2 = 0\,.$$

From (3) we obtain

$$u_1 = (y_2 y_3') = a_{11} y_1 + a_{12} y_2 + a_{13} y_3\,,$$
$$u_2 = (y_3 y_1') = a_{21} y_1 + a_{22} y_2 + a_{23} y_3\,,$$
$$u_3 = (y_1 y_2') = a_{31} y_1 + a_{32} y_2 + a_{33} y_3\,.$$

From u_1 and its first and second derivatives we obtain a_{11}, a_{12} and a_{13} by solving the equations

$$u_1 = (y_2 y_3') = a_{11} y_1 + a_{12} y_2 + a_{13} y_3\,,$$
$$u_1' = (y_2 y_3'') = a_{11} y_1' + a_{12} y_2' + a_{13} y_3'\,,$$
$$u_1'' = (y_2' y_3'') - 3P_2 (y_2 y_3') = a_{11} y_1'' + a_{12} y_2'' + a_{13} y_3''\,;$$

viz.

$$(1\,2'\,3'')\, a_{11} = 2\,(2'\,3'')\,(2\,3') - 3\,(2\,3')^2 P_2 - (2\,3'')^2\,,$$
$$(1\,2'\,3'')\, a_{12} = (2\,3')\,(3'\,1'') + (2'\,3'')\,(3\,1') - (2\,3'')\,(3\,1'') - 3P_2\,(2\,3')\,(3\,1')\,,$$
$$(1\,2'\,3'')\, a_{13} = (2\,3')\,(1'\,2'') + (2'\,3'')\,(1\,2') - (2\,3'')\,(1\,2'') - 3P_2\,(2\,3')\,(1\,2')\,.$$

As $(1\,2'\,3'')$ is constant, and also a_{11}, a_{12}, a_{13}, so must the right members of these equations be constant. If in these equations we interchange the suffixes the right members remain unchanged in value ; therefore

$$(1\,2'\,3'')\, a_{12} = (1\,2'\,3'')\, a_{21}\,, \quad a_{13} = a_{31}\,, \quad a_{23} = a_{32}\,.$$

Proceeding in a similar way for $n = 4$, we obtain

$$a_{11} = 4u_1^2\theta_3\,, \qquad a_{12} = (4'\,3'') - (4\,3''') - 6P_2(4\,3')\,,$$
$$a_{13} = (2'\,4'') - (2\,4''') - 6P_2(2\,4')\,, \quad a_{14} = (3'\,2'') - (3\,2''') - 6P_2(3\,2')\,.$$

Thus $a_{11} = 0$, as the invariant θ_3 is supposed to vanish.

Had we solved for a_{21}, a_{31}, and a_{41}, we should have found that $a_{21} = -a_{12}$, $a_{31} = -a_{13}$, and $a_{41} = -a_{14}$. That these results hold for all values of n will be seen by the following considerations :—

$$\frac{d\varphi}{dx} = \frac{\partial\varphi}{\partial y_1} y_1' + \frac{\partial\varphi}{\partial y_2} y_2' + \ldots + \frac{\partial\varphi}{\partial y_n} y_n' = 0\,,$$
$$\frac{d^2\varphi}{dx^2} = \frac{\partial\varphi}{\partial y_1} y_1'' + \frac{\partial\varphi}{\partial y_2} y_2'' + \ldots + \frac{\partial\varphi}{\partial y_n} y_n'' + 2\varphi(y') = 0\,;$$

but $\varphi(y')$ by equations (2) $= 0$.

Similarly we see that

$$\sum_{1}^{n}{}_{i} \frac{\partial \varphi}{\partial y_i} y_1^{(r)} = 0 , \qquad (r = 0, \ldots, n-2) \qquad (4)$$

which equations are similar to (1); and as the determinant formed with the y and their derivatives is not zero, $\frac{\partial \varphi}{\partial y_i}$ must be proportional to u_i; i. e.

$$\frac{\partial \varphi}{\partial y} = k u_i . \qquad (i = 1, 2, \ldots, n)$$

Then it follows that

$$2a_{11}y_1 + (a_{12} + a_{21}) y_2 + (a_{31} + a_{13}) y_3 + \ldots$$
$$+ (a_{1n} + a_{n1}) y_n = k (a_{11}y_1 + a_{12}y_2 + a_{13}y_3 + \ldots + a_{1n}y_n) ,$$
$$(a_{21} + a_{12}) y_1 + 2a_{22}y_2 + (a_{23} + a_{32}) y_3 + \ldots$$
$$+ (a_{2n} + a_{n2}) y_n = k (a_{21}y_1 + a_{22}y_2 + a_{23}y_3 + \ldots + a_{2n}y_n) ,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

As all the a_{ik}'s cannot vanish we must have $k = 0$ or 2. For $k = 0$,

$$a_{11} = a_{22} = a_{33} = a_{44} = = a_{nn} = 0$$

and

$$a_{ik} = - a_{ki} . \qquad (i = 1, 2, \ldots, n ; \; k = 1, 2, \ldots, n)$$

For $k = 2$,

$$a_{ik} = a_{ki} .$$

In order that the determinant of the a's may not vanish n must be even when $a_{ik} = - a_{ki}$. Thus we can announce the theorems,

When n is odd, the determinant of the a is symmetric.

When n is even, the determinant of the a is skew symmetric and the equation $\varphi(y)$ vanishes identically.

We then have to do with a linear complex, included in

$$u_1 y_1^{(l)} + u_2 y_2^{(l)} + \ldots + u_n y_n^{(l)} = 0 . \qquad (l = 1, 2, \ldots n-2) ,$$

or

$$a_{12} (1\,2^{(l)}) + a_{13} (1\,3_1^{(l)}) + \ldots + a_{1n} (1 n^{(l)}) + a_{23} (2\,3^{(l)}) + \ldots$$
$$+ a_{ik} (i k^{(l)}) + \ldots + a_{n-1,n} (n-1, n^{(l)}) = 0 . \quad (5)$$

Thus we see that the first associate of a self-adjoint equation is of order $\overline{\overline{<}}\, \frac{1}{2} n (n-1) - 1$.

When we take into consideration the relations existing between the associate variables, the application of the possible groups of substitutions shows that some of the coefficients a_{ik} may vanish.

By the introduction of a linear substitution we can cause all the a_{ik} to vanish except those for which $i + k = n + 1$ and that lie in the diagonal which is not principal, and these may have the value ± 1. For $n = 4$ (5) becomes

$$(y_1 y_4') + (y_2 y_3') = 0\,, \quad (y_1 y_n'') + (y_2 y_3'') = 0\,.$$

Taking $A = y^{iv} + 6P_2 y'' + 4P_3 y' + P_4 y = 0$, where $P_3 = \frac{3}{2} P_2'$, the solutions of the first associate will be

$$v_1 - (1\,2')\,,\ v_2 = (1\,3')\,,\ v_3 = (1\,4')\,,\ v_4 = (2\,4')\,,\ v_5 = (3\,4')\,.$$

Usually it is difficult to form the first associate, but the use of the values found for a_{ik} simplifies the process.

$$v = (1\,2')\,,$$

$$v^{\mathrm{I}} = (1\,2'')\,,$$

$$v^{\mathrm{II}} = 2(1'\,2'') - 6P_2 v + a_{34}\,,$$

$$v^{\mathrm{III}} + 6P_2 v' + 6P_2' v = 2(1'\,2''')\,,$$

$$v^{\mathrm{IV}} + 6P_2 v'' + 12P_2' v + 6P_2'' v = 2[(1''\,2''') + P_4 v' - 3P_2(v'' + 6P_2 u - a_{34})]\,,$$

$$A_1 = v^{\mathrm{V}} - 12\,P_2 v''' + 18P_2' v'' + (18P_2'' + 36P_2^2 - 4P_4)v'$$
$$+ (6P_2''' + 36P_2 P_2' - 2P_4')\,v = 0\,.$$

Forming the invariants θ_3 and θ_5 for this, we find that they vanish; A_1 is, therefore, self-adjoint.

It is not difficult to show that $A_1 = 0$ has for its first associate an equation of which the solutions are

$$y_1^2,\ y_1 y_2,\ y_1 y_3,\ y_1 y_4,\ y_2^2,\ y_2 y_3,\ y_2 y_4,\ y_3^2,\ y_3 y_4,\ y_4^2,\ \ldots\,.$$

After the substitution we have,

$$u_a \equiv (-1)^{a-1}(y_1 y_2' \ldots y_{a-1}^{(a-2)}\, y_{a+1}^{(a-1)} \ldots y_n^{(n-2)}) = a_{ak} y_k\,,$$

where $k = n + 1 - a$. If in any portion of the plane the y's have the form x^m, the exponents belonging to y_k being m_k, we obtain the following relations:

For $n = 3$,

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = a_{31} y_1 \text{ and } m_1 + m_2 - 1 = m_1\,; \text{ whence } m_2 = 1, \text{ and } m_1 + m_3 = 2\,.$$

For $n = 6$, we have

$(y_1 y_2^{\text{i}} y_3^{\text{ii}} y_4^{\text{iii}} y_5^{\text{iv}}) = a_k y_1$, then $m_1 + m_2 + m_3 + m_4 + m_5 - 10 = m_1$;

$(y_1 y_2^{\text{i}} y_3^{\text{ii}} y_4^{\text{iii}} y_6^{\text{iv}}) = c y_2$, and $m_1 + m_2 + m_3 + m_4 + m_6 - 10 = m_2$;

similarly, $m_1 + m_2 + m_3 + m_5 + m_6 - 10 = m_3$;

$$m_1 + m_2 + m_4 + m_5 + m_6 - 10 = m_4 .$$

Adding the first three and subtracting the fourth twice gives $2(m_3 + m_4) = 10$, and similarly for the rest; so that

$$m_1 + m_6 = m_2 + m_5 = m_3 + m_4 = 5 ,$$

and in like manner

$$m_1 + m_n = m_2 + m_{n-1} = m_3 + m_{n-2} = \ldots = m_s + m_{n-s+1} = n - 1 .$$

Where y_k be of the form $e^{m_k x}$, the same relations exist between the m, but $m_s + m_{n+1-s} = 0 \quad (s = 1, 2, 3, 4, \ldots)$.

In an interesting and exhaustive article in Crelle, Bd. 113, G. Wallenberg treats the class of equations known as Fuchsian equations. They have rational coefficients, the integrals behave regularly in the neighborhood of the singular points, and the roots of the indicial (*determinirende*, fundamental) equation are all rational numbers. Halphen has shown that an equation may be transformed into another with constant coefficients if the absolute invariants are constant.

Then for a self-adjoint equation having its absolute invariants constant and belonging to Fuchs' class. Mr. Wallenberg finds the y in the form

$$y_k = [R_{(x)}]^{-\frac{n-1}{4}} e^{-\gamma\mu_k \int \sqrt{R} dx} , \quad y_{n+1-k} = [R_{(x)}]^{-\frac{n-1}{4}} e^{\gamma\mu_k \int \sqrt{R} dx}$$

$$k = 1, 2, 3, \ldots .$$

Then there exists $n - 2$ homogeneous relations of the second order between the y; viz.:

$$y_1 y_n = y_2 y_{n-1} = y_3 y_{n-2} = \ldots = y_k y_{n+1-k} ,$$

$$\left[\frac{y_1}{y_n}\right]^{\mu_k} = \left[\frac{y_k}{y_{n+1-k}}\right]^{\mu_1} . \qquad (k = 1, 2, 3, \ldots)$$

In this $R(x)$ is a rational function which becomes unity so that

$$y_k = \frac{1}{y_{n+1-k}} = e^{-r_k x} .$$

Then we have

$$u_a = (-1)^{a-1} (y_1 y_2' \cdots y_{a-1}^{(a-2)}\, y_{a+1}^{(a-1)} \cdots y_n^{(n-2)}) .$$

Substituting the values of the y and taking $n = 6$,

$$u_2 = (-2)^{\frac{n}{2}-1} (-1)^{a-1} \frac{r_1 r_2 r_3}{r_2} \begin{vmatrix} 1 & 1 \\ r_1^2 & r_3^2 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ r_1^2 & r_2^2 & r_3^2 \\ r_1^4 & r_2^4 & r_3^4 \end{vmatrix} y_5 .$$

For $n = 7$,

$$u_2 = (-2)^3 (-1) \frac{r_1^3 r_2^3 r_3^3}{r_2^2} \begin{vmatrix} 1 & 1 \\ r_1^2 & r_3^2 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ r_1^2 & r_2^3 & r_3^2 \\ r_1^4 & r_2^4 & r_3^4 \end{vmatrix} y_6 ;$$

with similar expressions for the other adjoint variables. It is not difficult to find them for the equation of the nth order. They serve to verify the preceding theory.

When all the invariants vanish we have the case in which $A = 0$ may be reduced to the form $y^n = 0$, so that the solutions are

$$y_k = x^{k-1} ; \qquad (k = 1, 2, 3, \ldots, n)$$

and we have the parabolic relations

$$y_k^2 = y_{k-1}\, y_{k+1} . \qquad (k = 1, 2, \ldots, n-2)$$

THE CALCULUS OF VARIATIONS.

By Dr. Harris Hancock, Chicago, Ill.

Introduction and General Outline.

Since the time of the Bernouillis, mathematicians have in a greater or less degree considered problems which could be solved by methods of variations. Euler and Lagrange gave these methods more systematic and comprehensive forms, and founded the calculus of variations on a more scientific basis.

By extending these principles other mathematicians have augmented the subject in a wonderful manner; without, however, avoiding many difficulties which arise from want of rigor in the proofs, and from a misinterpretation of some of the fundamental conceptions. All these difficulties were removed when Prof. Karl Weierstrass, in 1879–'80, founded the entire calculus of variations on a new basis, free from any objection, and which at the same time is more comprehensive in its embrace.

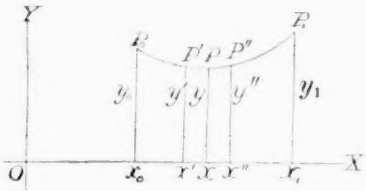
The writer does not think it out of place to bring these investigations before the readers of the Annals of Mathematics, since so little is known of this new treatment of the calculus of variations, especially to American students; and so through the courtesy of the editor he will give separate papers which are, so far as possible, complete in themselves, and which at the same time are intended to include the essential parts of the new theory of variations.

These papers are in a great measure abstracts of lectures that were given at Berlin by Prof. Weierstrass; much is also due to Prof. H. A. Schwarz, whose lectures on the calculus of variations the writer had the pleasure of hearing at Berlin during the summer semester of 1891.

1. *In the differential calculus* a definite function is given, and a special value of the variable or of the variables (if there are more than one variable) is sought, for which the function takes the greatest or the least possible value; in *the calculus of variations* a function is sought, and an expression which depends upon this function in a certain known manner is given. A definite integral is given, in which the integrand depends upon the unknown function in a known manner, and it is asked what form must the unknown function have in order that the definite integral may have a maximum or a minimum value.

We treat only real values of the variables.

2. In order to learn to recognize the general nature of the subject, we shall first state a few problems which may be solved by the calculus of variations; so that, while we seek the general characteristics of these problems, we shall of our own accord come to a more exact statement of the problems which the calculus of variations has to solve.



PROBLEM I. *Two points P_0 and P_1 with coordinates (x_0, y_0) and (x_1, y_1) respectively are given. Both points lie on the same side of the axis of X in the plane xy. It is required to join P_0 and P_1 by a curve such that when the plane is turned through one complete revolution about the axis of x, the zone generated by this curve may have the smallest possible surface.*

3. To show some of the defects of the old methods we proceed as follows: With the assumption that it is possible to draw a curve through the two points which satisfies the conditions of the problem, we suppose that two points $P'(x', y')$ and $P''(x'', y'')$ are taken on the curve, and we find another point $P(x, y)$ on the curve such that

$$x - x' = x'' - x = \Delta x .$$

We suppose that P and P', P and P'' are joined together by straight lines; and later we suppose that these three points are taken very close together, so that there is a transition from the two straight lines to the curve. The remaining portions of the curve on the left hand side of P' and on the right hand side of P'' are supposed to remain unaltered.

The portions of surface generated by the straight lines PP' and PP'' are, respectively,

$$\pi(y' + y)\sqrt{(\Delta x)^2 + (y - y')^2} \text{ and } \pi(y + y'')\sqrt{(\Delta x)^2 + (y'' - y)^2}.$$

The sum of these two surfaces of revolution we consider as a function of the variable y, and it is required to find when

$$\pi(y' + y)\sqrt{(\Delta x)^2 + (y - y')^2} + \pi(y + y'')\sqrt{(\Delta x)^2 + (y'' - y)^2}$$

is a minimum.

In order to have a minimum this expression when differentiated with regard to y must be zero; i. e.

$$\pi \sqrt{(\Delta x)^2 + (y - y')^2} + \pi \sqrt{(\Delta x)^2 + (y'' - y)^2}$$
$$+ \frac{\pi (y' + y)(y - y')}{\sqrt{(\Delta x)^2 + (y - y')^2}} - \frac{\pi (y + y'')(y'' - y)}{\sqrt{(\Delta x)^2 + (y'' - y)^2}} = 0 . \qquad \text{(A)}$$

y may be determined from this equation,

$$y = f(x), \text{ say.}$$

Therefore

$$y' = f(x - \Delta x),$$

and

$$y'' = f(x + \Delta x).$$

Hence, by Taylor's theorem,

$$y' = f(x - \Delta x) = f(x) - f'(x)\,\Delta x + \tfrac{1}{2} f''(x)\,(\Delta x)^2 - \dots,$$
$$y'' = f(x + \Delta x) = f(x) + f'(x)\,\Delta x + \tfrac{1}{2} f''(x)\,(\Delta x)^2 + \dots.$$

Hence

$$y - y' = f'(x)\,\Delta x - \tfrac{1}{2} f''(x)\,(\Delta x)^2 + \dots,$$
$$y'' - y = f'(x)\,\Delta x + \tfrac{1}{2} f''(x)\,(\Delta x)^2 + \dots.$$

Substituting these values in (A), we have

$$\Delta x \sqrt{1 + f'(x)^2 + \dots} + \Delta x \sqrt{1 + f'(x)^2 + \dots}$$
$$+ \frac{[2f(x) - f'(x)\,\Delta x + \dots]\,[f'(x)\,\Delta x - \tfrac{1}{2} f''(x)\,\Delta x^2 + \dots]}{\Delta x \sqrt{1 + f'(x)^2}}$$
$$- \frac{[2f(x) + f'(x)\,\Delta x + \dots]\,[f'(x)\,\Delta x + \tfrac{1}{2} f''(x)\,\Delta x^2 + \dots]}{\Delta x \sqrt{1 + f'(x)^2}} = 0 .$$

Dividing through by Δx and making $\Delta x = 0$, we have

$$1 + f'(x)^2 - f(x) f''(x) = 0 ; \qquad \text{(B)}$$

i. e.

$$1 + \left[\frac{dy}{dx}\right]^2 - y\frac{d^2y}{dx^2} = 0 .$$

Therefore in order to have a minimum value, $f(x)$ or y must satisfy this differential equation; however, when y satisfies this differential equation, we do not always have a minimum, as will be shown later.

Differentiate (B) with regard to x, and we have

$$\frac{dy}{dx} \cdot \frac{d^2y}{dx^2} = y\frac{d^3y}{dx^3},$$

or

$$\frac{\frac{dy}{dx}}{y} = \frac{\frac{d}{dx}\left[\frac{d^2y}{dx^2}\right]}{\frac{d^2y}{dx^2}}.$$

That is, $y = c^2 \dfrac{d^2y}{dx^2}$, where c^2 is the constant of integration.

Since $y = e^{\frac{x}{c}}$, and $y = e^{-\frac{x}{c}}$ are two solutions of this last differential equation, the most general solution is

$$y = c_1 e^{\frac{x}{c}} + c_2 e^{-\frac{x}{c}},$$

where c_1 and c_2 are also constants.

This last equation is the equation of the catenary.

4. Thus, by the help of the theory of maxima and minima, we have, it is true, come to a certain result; but, on the other hand, we have yet to ask whether this curve gives a true minimum, and owing to the manner in which we have come to the result, we have yet to see whether this curve only in a definite portion or throughout its whole extent possesses the property required in the problem.

That we are justified in insisting upon this last statement is seen from what follows later, where it will be shown that the curve found above satisfies the required conditions only between given limits.

A simple consideration shows that the method we have followed above is not at all rigorous; since it presupposes, which of itself is not admissible, that the curve which satisfies the problem is regular in its whole extent, since otherwise the portions of curve between the two points $(x - \Delta x, y')$ and (x, y) could not be replaced by straight lines joining these two points.

5. The characteristic difference between problems relative to maxima and minima and the problems which have to do with the calculus of variations, consists in the fact, that in the first case we have to deal with only a *finite number of discrete points*, while in the calculus of variations the question is concerning a *continuous series of points*.

If we wish to substitute in the place of the curve first a polygonal line and afterwards apply to this line methods similar to those used above, then it turns out that, after we have found a line which satisfies all the conditions, it is necessary yet to prove that the required limiting transition from polygonal

line to curve *in reality* results in a definite curve which satisfies the conditions of the problem.

The method given above has been chosen to make clear what is in common between, as well as the difference between, the theory of maxima and minima and the calculus of variations, and we shall now formulate the problem in a different manner.

6. Every limiting transition, as from polygon to curve, is made of itself, if we make use of the conception of integration ; since an integral represents the limiting value of a sum of quantities which increase following a definite law so as to become infinite in number, the quantities themselves becoming smaller and smaller in a corresponding manner.

If we, therefore, define the surface area of the curve $y = f(x)$ which we have to find by

$$S = 2\pi \int y ds\,,$$

or

$$\frac{S}{2\pi} = \int_{x_0}^{x_1} y \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx\,,$$

then this integral will have a definite value for every curve that is drawn between the points P_0 and P_1, and consequently the problem may be stated as follows :—

Problem I. *y is to be so determined as a function of x, that the above integral shall have the smallest possible value.*

The solution of this problem will be given later.

7. As a second problem may be given the problem of the *brachistochrone* (curve of quickest descent) which may be stated as follows :—

Problem II. *In a vertical plane a curve is to be drawn from a point A to a point B below in such a manner that a material point which is acted upon by gravity, and which is compelled to move upon this curve, shall with a given initial velocity go from A to B in the shortest possible time.*

Let the mass of the material point be 1, its initial velocity a, the acceleration of gravity $2g$, the time t, and the coordinates of A and B respectively $(0, 0)$ and (a, b).

Let the direction of the positive y-axis be the direction of a falling body (gravity), and let the positive x-axis be directed towards the side on which B lies. Then according to the law of the *conservation of energy*,

$$\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 = 4gy + a^2$$

or

$$dt = \frac{\sqrt{dx^2 + dy^2}}{\sqrt{4gy + a^2}} = \frac{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy}{\sqrt{4gy + a^2}}\,;$$

whence

$$T = \int_0^b \frac{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}}{\sqrt{4gy + a^2}}\, dy\,.$$

Our problem then is to so determine x as a function of y that the above integral shall have the smallest possible value.

8. In the two problems given above one of the variables is a one-valued function of the other; this is due to the fact that the system of coordinates may be so chosen in both cases. Since this is not possible in all cases, it is expedient to represent the curve by two equations, that is, to consider x and y as one-valued functions of any quantity t, where t has only the property, that when it goes through all values between two given limits, the corresponding point x, y traverses the curve from the beginning point to the end point, and in such a way that to a greater value of t there corresponds a later* point of the curve. Hence the integrals of our two problems, which are to have a minimum value may be expressed in the form

$$S = 2\pi \int_{t_0}^{t_1} y \sqrt{x'^2 + y'^2}\, dt\,, \tag{I}$$

$$T = \int_{t_0}^{t_1} \frac{\sqrt{x'^2 + y'^2}}{\sqrt{4gy + a^2}}\, dt\,, \tag{II}$$

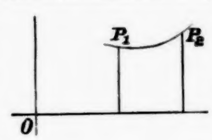
where

$$x' = \frac{dx}{dt}\,, \quad y' = \frac{dy}{dt}\,.$$

9. Problem III. *Between two points on a regular surface $f(x, y, z) = 0$, a curve is to be drawn so that its length is a minimum.*

Consider the orthogonal coordinates x, y, z of a point of the surface represented as one-valued regular functions of two parameters u, v. If we consider these as the rectangular coordinates of a point of the plane, then to every

* If $t_1 < t_2$ and the point P_1 corresponds to t_1, and P_2 to t_2, then P_2 in reference to P_1 is known as a *later* point, and P_1 in reference to P_2 as an *earlier* point.



point of the surface there will correspond a definite point of the uv-plane, and these points in their collectivity fill out a definite part of the plane, which may be looked upon as the image of the surface on the plane. To every curve on the surface corresponds a curve in this part of the uv-plane and reciprocally.

Consider, further, u and v as onevalued functions of a quantity t; hence to every value t there corresponds a point of the uv-plane, and therefore, also, in case this point lies in the definite part of the uv-plane, there is a corresponding definite point of the surface.

Consequently if t_0 and t_1 are values of t which correspond to the two fixed points on the surface, then the length of any curve which lies between these two points is determined through

$$L = \int_{t_0}^{t_1} \sqrt{P\left[\frac{du}{dt}\right]^2 + 2Q\frac{du}{dt}\cdot\frac{dv}{dt} + R\left[\frac{dv}{dt}\right]^2}\, dt,$$

where

$$P = \left[\frac{\partial x}{\partial u}\right]^2 + \left[\frac{\partial y}{\partial u}\right]^2 + \left[\frac{\partial z}{\partial u}\right]^2,$$

$$Q = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v},$$

$$R = \left[\frac{\partial x}{\partial v}\right]^2 + \left[\frac{\partial y}{\partial v}\right]^2 + \left[\frac{\partial z}{\partial v}\right]^2.$$

We have to determine u and v as functions of t, so that L is a minimum.

10. What is common to these three problems is that we have to determine two functions of t in such a way that an integral depending upon them and of the form

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt$$

may have the smallest possible value.

Here t_0 and t_1 have fixed values, and $F(x, y, x', y')$ represents a function of x, y, x', y', of which x', y' may be regarded as unlimited variables, while x, y are limited to a region which extends over the whole plane, or over a continuous part of the same.

11. The condition that t_0, t_1 should have fixed values is not essential; moreover both end points may move, as in the case of the third problem, if we give it the following form: Two curves are given on a surface; among all the possible curves between the points of the one curve and the points of the other curve, that curve is to be found which has the shortest length. We are accustomed to call this the geodesic distance of the two curves.

In order to solve this problem, we must first solve the special problem III, since if a curve has the property of being of minimum length such as is required above, it must also retain the same property, if we consider the end points fixed. Hence, from III the nature of the curve must be determined. The variation of the end points give in addition certain special properties, which the curve must possess.

For example, the shortest distance between two curves which lie in the same plane is clearly a straight line; through the variation of the end points it follows that this straight line must be at the same time perpendicular to both curves.

12. PROBLEM IV. Essentially different from the three problems given above is the following: *We are so to construct a closed curve with given periphery, that the surface inclosed shall have the greatest possible area.*

Consider x and y such functions of t, say $x(t)$, $y(t)$, that for two definite values t_0 and t_1 of t, the corresponding points x, y of the curve fall together, and that, if t goes from a smaller value t_0 to a greater value t_1, x, y completely traverses the curve in the positive direction; then twice the area of the surface inclosed by the curve will be expressed by the integral

$$I = \int_{t_0}^{t_1} (xy' - yx')\, dt\,,$$

and the periphery of the curve is determined by means of

$$I_1 = \int_{t_0}^{t_1} \sqrt{x'^2 + y'^2}\, dt\,.$$

13. The proposed problem is now as follows: *So determine x, y as functions of t, that the integral I which depends upon them, shall have the greatest possible value, while at the same time I_1 has a given fixed value.*

Problems of this nature are the most interesting and of the most frequent occurrence. They require a treatment essentially different from that of those first mentioned.

These problems are sufficient to give a conception of the nature of the problems which are to be solved by the calculus of variations, and with these as a basis it will be possible to define the object of the calculus of variations.

We must yet, however, introduce the fundamental conception of the variation of a curve. In former times the calculus of variations was considered one of the most difficult branches of analysis; it was wrongly thought that the ground of this difficulty was in the supposed lack of clearness in the fundamental conceptions, especially in the conception of the variation of a curve,

while the difficulties that do arise, lie for the most part in quite another direction.

14. In the theory of maxima and minima we say the value of a function is, for a definite system of values of the variables, a maximum or a minimum, if this value of the function for this system of values is greater or smaller than for all the neighboring systems of values.

We say of a function $f(x)$ of one variable, it has at a definite position $x = a$, a maximum or a minimum value, if the value for $x = a$ is respectively greater or less than it is for all other values of x which are situated in the neighborhood $|x - a| < \delta$ as near as we wish to a.

The analytical condition that $f(x)$ shall have for the position $x = a$

$$\left.\begin{array}{l} a \textit{ maximum} \text{ is expressed by } f(x) - f(a) < 0\,; \\ a \textit{ minimum} \text{ “ \quad “ \quad “ } f(x) - f(a) > 0\,. \end{array}\right\} |x - a| < \delta\,.$$

In the same way, *we say of a function $f(x_1, x_2, \ldots, x_n)$ of n variables, that it has for a definite position $x_1 = a_1, x_2 = a_2, \ldots, x_n = a_n$, a maximum or a minimum, if the value of the function for $x_1 = a_1, x_2 = a_2, \ldots, x_n = a_n$ is respectively greater or smaller than it is for all other systems of values, which are situated in the neighborhood $|x_\lambda - a_\lambda| < \delta_\lambda$ ($\lambda = 1, 2, \ldots, n$), as near as we wish to the first position.*

As here we speak of a neighboring system of values, so also in the calculus of variations, we speak of curves which lie in the neighborhood of a given curve; and we require that an integral in the case of a minimum should be less and in the case of a maximum greater when taken over the given curve than for any of the neighboring curves.

15. In order to fix the conception of a neighboring curve, and to make clear the analogy of the same with the conception of a neighboring system of values, let us consider first instead of the given curve a broken line $A_1A_2A_3 \ldots A_n$, and let us cause the same to slide just a little from its original position.

Then in the new position every corner B_k will correspond to a definite corner A_k in the old position, and moreover the new position $B_1B_2B_3 \ldots B_n$ will be as little different from the old position $A_1A_2A_3 \ldots A_n$ as we wish, if we stipulate that the distance between any two corresponding points A_k and B_k shall be smaller than any quantity δ where δ is as small as we choose. Now, by increasing the number of sides, let the broken line pass into the given curve, then the points $B_1B_2 \ldots B_n$ will also form a curve which is little different from the first curve, and which we consequently call *neighboring to* the first curve.

We can, therefore, say a curve *is neighboring to* another curve, or exists

out of another curve through a variation as small as we choose, if to every point of the latter curve there corresponds a definite point on the former curve and also the distance between any two corresponding points is smaller than δ, where δ is as small as we choose.

This geometrical conception of a neighboring curve offers no obscurity. In a similar manner it is easy to see that for every change of the curve, there is a corresponding change of the integral

$$\int F(x, y, x', y')\, dt\,,$$

and that this change will be infinitely small, when the second curve is neighboring to the first.

This change of the value of the integral must of course be a continuous negative one, if the integral is to be a maximum, and a continuous positive one, if the integral is to be a minimum.

16. In accordance with this we may formulate the problem of the calculus of variations as follows :

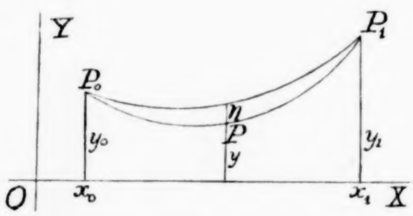
The quantities x, y are to be so determined as functions of a quantity t, that when we define a curve by the equations $x = x(t)$, $y = y(t)$, and cause the curve to vary as little as we choose, the change which in consequence takes place in the integral

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt$$

is continuously positive if a minimum, and continuously negative if a maximum is to enter.

If we consider the problem as proposed in this manner, we have a definite problem of the calculus of variations before us, and we have to find strict and rigorous methods for the solution of this problem.

17. *Variation of curves.*



Let us return for a moment to the integral

$$S = \int_{x_0}^{x_1} 2\pi y \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx\,. \tag{1}$$

Let $y = f(x)$ be the curve which gives a minimum surface area when we rotate this curve about the axis of x.

Let η be the distance between this curve and any neighboring curve measured on the y-ordinate, and suppose that η is a continuous function of x subject to the conditions, that for $x = x_0$, $\eta = 0$; for $x = x_1$, $\eta = 0$; and for all other points $|\eta| < \rho$, where ρ may be as small as we choose.

$$\eta' = \frac{d\eta}{dx} \text{ and } \int_{x_0}^{x_1} \eta' dx = [\eta]_{x_0}^{x_1} = 0 .$$

The integral of any neighboring curve corresponding to (1) is

$$\int_{x_0}^{x_1} 2\pi (y + \eta) \sqrt{1 + \left[\frac{d(y+\eta)}{dx}\right]^2} dx . \tag{2}$$

Hence the total variation caused in (1) when, instead of $y = f(x)$, we take a neighboring curve, is

$$\varDelta S = \int_{x_0}^{x_1} 2\pi (y + \eta) \sqrt{1 + \left[\frac{d(y+\eta)}{dx}\right]^2} dx - \int_{x_0}^{x_1} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx . \tag{3}$$

$\varDelta S$ has always a positive sign since the surface in question is a minimum.

18. Instead of the one neighboring curve, we may consider a whole bundle of such curves, if for η we substitute $\varepsilon\eta$, where ε is independent of x and has any value between -1 and $+1$. After this substitution (3) becomes

$$\varDelta S = \pi \left[\int_{x_0}^{x_1} 2 (y + \varepsilon\eta) \sqrt{1 + \left[\frac{d}{dx}(y + \varepsilon\eta)\right]^2} dx - \int_{x_0}^{x_1} 2y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \right]; \tag{4}$$

and, on the other hand, developing $\varDelta S$ by Taylor's theorem,

$$\varDelta S = \varepsilon \delta S + \frac{\varepsilon^2}{1 . 2} \delta^2 S + \frac{\varepsilon^3}{1 . 2 . 3} \delta^3 S + \dots . \tag{5}$$

There is no constant term in this last development, since when ε is made zero in (4) the first and second integrals cancel each other.

δS is the first variation,

$\delta^2 S$ is the second variation, etc.

Instead of taking η a very small quantity, we may take ε so small that $\varepsilon\eta$ is as small as we choose.

With Lagrange, writing $\eta = \delta y$, it is seen that the total change in y is $\varepsilon\eta = \varepsilon\delta y = \Delta y$.

Remark. The sign of differentiation and the sign of variation may be interchanged; for example the 1st derivative of a variation is equal to the 1st variation of a derivative, as is seen by writing

$$\eta = \delta y, \text{ then } \eta' = (\delta y)' = \frac{d}{dx}(\delta y). \qquad (1)$$

Again $\eta = \delta y$; change y into $y + \varepsilon\eta$, and consequently y' into $y' + \varepsilon\eta'$; whence

$$\eta' = \delta y' = \delta\left[\frac{dy}{dx}\right]; \qquad (2)$$

hence, from (1) and (2),

$$\frac{d}{dx}(\delta y) = \delta\left[\frac{dy}{dx}\right].$$

19. Returning to (4) and writing $y' = \frac{dy}{dx}$, $\eta' = \frac{d\eta}{dx}$ and expanding the expression under the sign of integration

$$2\pi(y + \varepsilon\eta)\sqrt{1 + (y' + \varepsilon\eta')^2} - 2\pi y\sqrt{1 + y'^2},$$

we have

$$\pi\varepsilon\left[2\eta\sqrt{1 + (y' + \varepsilon\eta')^2} + \frac{(y + \varepsilon\eta)(y' + \varepsilon\eta')\eta'}{\sqrt{1 + (y' + \varepsilon\eta')^2}}\right]_{\varepsilon=0} + \varepsilon^2(\ldots).$$

Hence, equating the coefficients of the 1st power of ε in (4) and in (5) we have

$$\delta S = 2\pi\int_{x_0}^{x_1}\left[\sqrt{1 + y'^2}\cdot\eta + \frac{yy'}{\sqrt{1 + y'^2}}\eta'\right]dx,$$

which is a homogeneous function of the 1st degree in η and η' (η' cannot be infinitely large, since then the development would not be necessarily convergent).

In a similar manner we may find a definite integral for the 2nd variation, in which the integrand is an integral homogeneous function of the 2nd degree in η and η'; similarly for the third variation, etc.

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